SHORT COMMUNICATION

COMMENT ON 'MODIFICATION OF THE SMAC METHOD IN TWO DIMENSIONS'[1]

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The author of Reference 1 presents equations (1)–(5) to replace the SMAC method for the solution of Navier–Stokes equations in two dimensions. The purpose of this comment is to show that the analysis of Reference 1 is *exactly* the ω–ψ formulation without any modification.

Let us start with the ω–ψ method. The continuity equation is identically satisfied using the following relations.

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{C1}$$

The vorticity definition $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$, becomes

$$\nabla^2 \psi = -\omega \tag{C2}$$

where the vorticity ω is calculated from the momentum equations after eliminating the pressure using cross differentiation, thus:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{C3}$$

Equations (C1) and (C2) are used in Reference 1 as equations (5) and (4)*, respectively. Then the vorticity ω is calculated from equations (1)–(3) instead of Equation (C3) as in the ω–ψ method. We show in the following section that Equations (1)–(3) in Reference 1 are *exactly* Equation (C3).

Substituting equations (1) and (2) into equation (3), one obtains

$$\begin{aligned} \omega^{n+1} = \left(\frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}\right) + \Delta t\bigg[& -u^n \frac{\partial}{\partial x}\left(\frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}\right) \\ & - v^n \frac{\partial}{\partial y}\left(\frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}\right) - \frac{\partial u^n}{\partial x}\left(\frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}\right) - \frac{\partial v^n}{\partial y}\left(\frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}\right) \\ & + \nu\frac{\partial^2}{\partial x^2}\left(\frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}\right) + \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}\right)\bigg] \end{aligned} \tag{C4}$$

* There is a sign difference which is most probably a typing mistake in Reference 1.

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Since

$$\omega^n = \frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y}$$

equation (C4) can be written as follows

$$\omega^{n+1} = \omega^n + \Delta t\left[-u^n\frac{\partial \omega^n}{\partial x} - v^n\frac{\partial \omega^n}{\partial y} + \nu\left(\frac{\partial^2\omega^n}{\partial x^2} + \frac{\partial^2\omega^n}{\partial y^2}\right)\right] \tag{C5}$$

which is exactly equation (C3).

Furthermore, the author of Reference 1 argues that his formulation using equations (1)–(3) did not have to approximate boundary conditions for ω. In fact he does just this in the solution of equations (1) and (2). He calculates the terms $\frac{\partial v^n}{\partial x} - \frac{\partial u^n}{\partial y} = \omega^n$ at the solid boundaries, which are nothing but the vorticity boundary conditions. This is shown as follows.

In the ω–ψ method, Dirichlet boundary conditions for ω are calculated by applying $\nabla^2\psi = -\omega$ on the boundary. For example at $x = 0$, $\psi(0, y) = 0$ and

$$\omega(0, y) = -\nabla^2\psi = -\frac{\partial^2\psi}{\partial x^2} \tag{C6}$$

By using Taylor's expansion and satisfying the no-slip condition, $\left(v = 0, \frac{\partial\psi}{\partial x} = 0\right)$,[2] equation (C6) can be written as

$$\omega(0, y) = -\frac{2}{h^2}\psi(h, y) + O(h) \tag{C7}$$

where h is the spatial step.

In the modified SMAC method, the terms $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega$ are calculated on the boundary and used in the solution of equations (1) and (2). At $x = 0$, $u = 0\left(\frac{\partial u}{\partial y} = 0\right)$; upon substitution in $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega$, one obtains

$$\omega(0, y) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{C8}$$

Since $v = -\frac{\partial\psi}{\partial x}$, equation (C8) becomes

$$\omega(0, y) = -\frac{\partial^2\psi}{\partial x^2} \tag{C9}$$

Equation (C9) is identical to equation (C6) and leads to the same Dirichlet boundary conditions for ω given by equation (C7).

It is clear that the two methods, ω–ψ and the modified SMAC, solve the same vorticity transport equation (C3) with the same Dirichlet boundary conditions equation (C7). In addition they solve the same Poisson equation (C2) for ψ with the same boundary conditions ($\psi = 0$): thus the numerical results of both methods must be identical.

REFERENCES

1. Wlodzimierz Kordylewski, 'Modification of the SMAC method in two dimensions', *Int. j. numer. methods fluids*, **2,** 407–409 (1982).
2. P. J. Roache, *Computational Fluid Dynamics*, Hermosa Publishers, 1976.